Noether's Theorems and Gauge Symmetries

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Consideration of the Noether variational problem for any theory whose action is invariant under global and/or local gauge transformations leads to three distinct theorems. These include the familiar Noether theorem, but also two equally important but much less well-known results. We present, in a general form, all the main results relating to the Noether variational problem for gauge theories, and we show the relationships between them. These results hold for both Abelian and non-Abelian gauge theories.

1 Introduction

There is widespread confusion over the role of Noether's theorem in the case of local gauge symmetries,[1] as pointed out in this journal by Karatas and Kowalski (1990), and Al-Kuwari and Taha (1991).[2] In our opinion, the main reason for the confusion is failure to appreciate that Noether offered two theorems in her 1918 work. One theorem applies to symmetries associated with finite dimensional Lie groups (global symmetries), and the other to symmetries associated with infinite dimensional Lie groups (local symmetries); the latter theorem has been widely forgotten. Knowledge of Noether's 'second theorem' helps to clarify the significance of the results offered by Al-Kuwari and Taha for local gauge symmetries, along with other important and related work such as that of Bergmann (1949), Trautman (1962), Utiyama (1956, 1959), and Weyl (1918, 1928/9). In this paper we present all the key results concerning Noether's theorems for global and local gauge symmetries - including those which go beyond Noether's own derivations - in the form of three simple theorems and their consequences.

[1] The confusion goes beyond gauge symmetries; for discussion in this journal of this point see, for example, Munoz (1996). The results presented in this paper extend straightforwardly to such cases.

[2] A key issue addressed in these papers is why no further conserved quantities arise from local gauge symmetries than already arise from global gauge symmetries. The reason for this is clearly seen in what follows.

In the process, we highlight several important and useful results that have been largely overlooked, and aim to help bring an end to the confusion over this issue.

The results we present are all derivable from the variational problem stated in section 2. Section 3 considers global gauge symmetry, and states the associated (and familiar) Noether theorem. Sections 4-6 address local gauge symmetry. In section 4 we state the second Noether theorem, and give an example of its applications. Section 5 addresses the application of the first Noether theorem to global subgroups of local gauge groups. Finally, in section 6, we discuss the paper of Al-Kuwari and Taha (1991). Their paper is based on results due to Utiyama (1956); we summarise these in the form of a theorem, and highlight what we what believe to be the most important aspect of Al-Kuwari and Taha's paper.

2 Basis of the Noether Theorems

It is useful to compare Noether's variational *problem* with the more familiar Hamilton's *principle.*

Consider a Lagrangian density L depending on N distinct fields ψ_i ($i = 1, ..., N$) and their first derivatives, written as $L = L(\psi_i, \partial_\mu \psi_i, x^\mu)$.[3] The action S is defined as $S = \int L d^4x$ over some compact region of space-time. Hamilton's principle, defined for a particular field, ψ_k, requires the action to be extremal (that is, $\delta S = 0$, where δS is the first order functional variation in S) for *arbitrary* variations of ψ_k which vanish on the boundary. As is well-known, the necessary and sufficient condition for this principle to hold is satisfaction of the Euler-Lagrange equations for ψ_k. (Note that the principle may not apply to all the fields on which a given Lagrangian depends. For an example, see section 6, footnote 14.)

Noether's variational problem (VP) can be posed as follows:

What general conditions must hold in order that a given variation of the dependent and/or independent variables leaves the action invariant, and hence $\delta S = 0$, *where* δS *may now contain a boundary term?*

Clearly this variational problem is importantly different from Hamilton's principle, both in the sets of variations considered, and in purpose.

[3] The restriction of L to $L = L(\psi_i, \partial_\mu \psi_i, x^\mu)$ and no higher derivatives of ψ_i is for convenience. The generalisation of everything that follows to higher derivatives is straightforward.

The general solution of VP is the following condition:[4]

$$\sum_i [\Psi]_i \, \delta_0 \psi_i \equiv - \sum_i \partial_\mu B_i^\mu \tag{1}$$

where

1. $[\Psi]_i$ is the 'Lagrange expression' associated with the field ψ_i:

$$[\Psi]_i := \frac{\partial L}{\partial \psi_i} - \partial_\mu \left(\frac{\partial L}{\partial \left(\partial_\mu \psi_i \right)} \right) \tag{2}$$

   i.e., $[\Psi]_i = 0$ are the Euler-Lagrange equations for ψ_i;

2. the variation of each ψ_i (denoted by $\delta\psi_i$) is composed of the direct variation in ψ_i plus that which arises as a consequence of the variation in x^μ:

$$\delta\psi_i = \delta_0 \psi_i + \left(\partial_\mu \psi_i \right) \delta x^\mu; \tag{3}$$

   and

3. the form of B_i^μ is:

$$B_i^\mu := \left(L \delta x^\mu + \frac{\partial L}{\partial \left(\partial_\mu \psi_i \right)} \delta_0 \psi_i \right). \tag{4}$$

Throughout this paper, we use the symbol '$\equiv$' to indicate equations that are derived without making use of any Euler-Lagrange equations, and the Einstein convention to sum over repeated Greek indices, all other summations being explicit.

In the case of gauge transformations in field theory we are concerned with transformations of the fields only (i.e., the dependent variables), and not transformations of the space-time coordinates (the independent variables), and hence we ignore terms in δx^μ. In this case, we have $\delta_0 \psi_i = \delta \psi_i$ and (1) becomes

$$\sum_i [\Psi]_i \, \delta \psi_i \equiv - \sum_i \partial_\mu C_i^\mu \tag{5}$$

where

$$C_i^\mu := \frac{\partial L}{\partial \left(\partial_\mu \psi_i \right)} \delta \psi_i. \tag{6}$$

[4]Details of the derivation can be found in Noether (1918), Doughty (1990, p. 338) and Brading and Brown (2000), for example. Note that VP, and the resulting expression, may be generalised to allow that the Lagrangian may pick up a divergence term under the variation. This is needed for Galilean boosts in particle mechanics, for example. For further discussion of this point see Doughty (1990) and Brading and Brown (2000).

This is the first stage in the derivation of all three theorems presented in this paper.[5]

When the Euler-Lagrange field equations are satisfied for all of the fields on which the Lagrangian depends, the 'Lagrange expressions' $[\Psi]_i$ vanish, so $\sum_i [\Psi]_i = 0$,[6] and from (5) we have the continuity equation

$$\sum_i \partial_\mu C_i^\mu = 0. \tag{7}$$

This result is sometimes referred to as 'Noether's theorem'. As is well known, 'Noether's theorem' is used to connect symmetries with conserved currents (and thence conserved charges, subject to suitable boundary conditions).[7] Confusion can arise when we attempt to use (7) to form conserved currents associated with local gauge symmetries, as we will see in what follows.[8]

3 Global Gauge Symmetry: Noether's First Theorem

In the case where the action S ($S = \int L d^4x$) is invariant under a finite dimensional continuous group of transformations depending smoothly on ρ independent parameters ω_α, ($\alpha = 1, 2, \ldots , \rho$), i.e. when the symmetry is global, we can write

$$\delta\psi_i = \sum_\alpha \frac{\partial\left(\delta\psi_i\right)}{\partial\left(\Delta\omega_\alpha\right)}\Delta\omega_\alpha \tag{8}$$

where $\Delta\omega_\alpha$ is used to indicate that we take infinitesimal ω_α. We substitute this into (5), yielding

$$\sum_i [\Psi]_i \frac{\partial\left(\delta\psi_i\right)}{\partial\left(\Delta\omega_\alpha\right)}\Delta\omega_\alpha \equiv -\sum_i \partial_\mu\left(\frac{\partial L}{\partial\left(\partial_\mu\psi_i\right)}\frac{\partial\left(\delta\psi_i\right)}{\partial\left(\Delta\omega_\alpha\right)}\Delta\omega_\alpha\right). \tag{9}$$

Then, since $\Delta\omega_\alpha$ is not a function of space or time, it can be removed from the derivative on the right-hand side and cancelled. This completes the derivation of **Noether's First Theorem**, which we now state.

[5] Generalisations of all these theorems, based on (1) rather than (5), are straightforward. See Brading and Brown, 2000.

[6] The assumption that the Euler-Lagrange field equations are satisfied yields $\sum_i [\Psi]_i = 0$ *iff* all the fields on which the Lagrangian depends satisfy Euler-Lagrange equations. The significance of this remark will be made clear in section 6, below.

[7] For an excellent discussion of the connection between a transformation having the status of a symmetry, in the sense of preserving the form of the Euler-Lagrange equations, and the invariance of the action under the transformation, see Doughty (1990, sections 9.2 and 9.5).

[8] Problems can also arise with respect to space-time transformations when $\sum_i \partial_\mu B_i^\mu = 0$ is used to form a conserved current. See, for example, Munoz (1996). See also Brading and Brown (2000) for further discussion.

Theorem 1 *If the action S is invariant under a finite dimensional continuous group of transformations depending smoothly on ρ independent parameters ω_α, $(\alpha = 1, 2, ..., \rho)$, then there exist the ρ relationships*

$$\sum_i [\Psi]_i \frac{\partial (\delta_0 \psi_i)}{\partial (\Delta\omega_\alpha)} \equiv \partial_\mu j^\mu_\alpha \tag{10}$$

where

$$j^\mu_\alpha = -\sum_i \frac{\partial L}{\partial (\partial_\mu \psi_i)} \frac{\partial (\delta \psi_i)}{\partial (\Delta\omega_\alpha)}. \tag{11}$$

When the Euler-Lagrange field equations are assumed to be satisfied for all the fields on which the Lagrangian depends, it follows from Noether's First Theorem that there exist ρ conserved currents, one for every parameter on which the symmetry group depends:

$$\partial_\mu j^\mu_\alpha = 0. \tag{12}$$

Subject to suitable boundary conditions, this may be integrated to give a conserved charge:

$$\frac{d}{dt} Q_\alpha = 0 \tag{13}$$

where

$$Q_\alpha := \int d^3 x j^0_\alpha (x). \tag{14}$$

Clearly, however, if $\Delta\omega_\alpha$ is an arbitrary function of x^μ rather than a constant, it cannot be eliminated from (7) in this way to give a current that is independent of the gauge-parameter, and this is where the potential confusions begin to arise. We consider the case of gauge symmetries depending on arbitrary functions of x^μ in the following sections.

For a concrete example of Noether's First Theorem, consider the global gauge symmetry of the Lagrangian associated with the Klein-Gordon equation for a free complex scalar field:

$$L_m = \partial_\mu \psi \partial^\mu \psi^* - m^2 \psi \psi^*. \tag{15}$$

L_m is invariant under $\psi \to \psi' = \psi e^{i\theta}$, $\psi^* \to \psi^{*\prime} = \psi^* e^{-i\theta}$, θ a constant, and the corresponding conserved Noether current is

$$j^\mu_{L_m} = i \left(\psi^* \partial^\mu \psi - \psi \partial^\mu \psi^* \right). \tag{16}$$

This application of Noether's First Theorem to global gauge symmetry is entirely familiar (see for example Ryder, 1996, p. 91).

4 Local Gauge Symmetry: Noether's Second Theorem

Consider now an infinite dimensional group of transformations depending smoothly on ρ arbitrary functions $p_\alpha(x^\mu)$ ($\alpha = 1, 2, \ldots, \rho$) and their first derivatives, and denote such a group by $G_{\infty\rho}$.[9] For an infinitesimal transformation of ψ_i we can write

$$\delta\psi_i = \sum_\alpha \left\{ a_{\alpha i}\left(\psi_i, \partial_\mu\psi_i, x^\mu\right)\Delta p_\alpha(x^\mu) + b^\mu_{\alpha i}\left(\psi_i, \partial_\mu\psi_i, x^\mu\right)\partial_\mu\left(\Delta p_\alpha(x^\mu)\right)\right\} \quad (17)$$

where the Δp_α indicates that we are taking infinitesimal p_α. We can then make use of (5) to prove the following theorem, found in Noether's 1918 paper, which we will refer to as **Noether's Second Theorem**.[10]

Theorem 2 *If the action S is invariant under a group $G_{\infty\rho}$ then there exist the ρ relationships*

$$\sum_i [\Psi]_i \, a_{\alpha i} \equiv \sum_i \partial_\mu \left([\Psi]_i \, b^\mu_{\alpha i}\right). \quad (18)$$

This is derived by noticing that, since they are arbitrary, we could choose the p_α and their derivatives so that they vanish on the boundary. Thus, the interior contribution to VP must vanish independently of the boundary contribution, and (18) is the condition for the vanishing of the integral associated with the interior contribution. The theorem tells us that there are dependencies between the Lagrange expressions $[\Psi]_i$ and their derivatives. This dependency follows from the local gauge invariance of the Lagrangian, and its precise form depends on the particular structure of the gauge transformation.[11]

To make the content of this theorem concrete, consider the specific case

$$L = D_\mu\psi D^\mu\psi^* - m^2\psi\psi^* - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \quad (19)$$

where $D_\mu = \partial_\mu + iqA_\mu$ is the covariant derivative, and $F^{\mu\nu}$ is some function of $\partial^\nu A^\mu$ but not of A^μ. L is invariant under local gauge transformations

$$\left.\begin{array}{c} \psi \to \psi' = \psi e^{iq\theta(x)} \\ \psi^* \to \psi^{*\prime} = \psi^* e^{-iq\theta(x)} \\ A_\mu \to A'_\mu = A_\mu + \partial_\mu\theta(x). \end{array}\right\} \quad (20)$$

[9] The restriction to the first derivative is again imposed for convenience, and the results presented here extend straightforwardly to include higher derivatives.

[10] For details of the derivation see Noether, 1918; Trautman, 1962; Brading and Brown, 2000.

[11] When the Lagrangian depends on a single field, the Second Theorem leads to a contraint on the Lagrange expression. Consider, for example, classical Maxwell electromagnetism (see Brading, 2000, for details).

In this case, we have only one arbitrary function $p = \theta$, and, infinitesimally,

$$\left.\begin{array}{c} \delta\psi = iq(\Delta\theta)\psi \\ \delta\psi^* = -iq(\Delta\theta)\psi^* \\ \delta A_\mu = \partial_\mu(\Delta\theta). \end{array}\right\} \tag{21}$$

Hence, from (17) we see that

$$\left.\begin{array}{c} a_\psi = iq\psi,\, b^\nu_\psi = 0 \\ a_{\psi^*} = -iq\psi^*,\, b^\nu_{\psi^*} = 0 \\ a_{A_\mu} = 0,\, b^\nu_{A_\mu} = \delta^\nu_\mu. \end{array}\right\} \tag{22}$$

Therefore (18) yields

$$\begin{aligned} &\left[\frac{\partial L}{\partial \psi} - \partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu \psi)}\right)\right] iq\psi + \left[\frac{\partial L}{\partial \psi^*} - \partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu \psi^*)}\right)\right](-iq\psi^*) \\ \equiv\;\; &\partial_\mu\left[\frac{\partial L}{\partial A_\mu} - \partial_\nu\left(\frac{\partial L}{\partial(\partial_\nu A_\mu)}\right)\right], \end{aligned} \tag{23}$$

from which we conclude that

$$\partial_\mu\partial_\nu F^{\mu\nu} \equiv 0 \tag{24}$$

where we have defined $F^{\mu\nu}$ as

$$F^{\mu\nu} := \frac{\partial L}{\partial(\partial_\nu A_\mu)}. \tag{25}$$

Equation (24) states that the derivative $\partial_\nu A_\mu$ must appear in the Lagrangian in the combination $\partial_\nu A_\mu - \partial_\mu A_\nu$, making $F^{\mu\nu}$ anti-symmetric.

In the few places where Noether's Second Theorem is discussed, the above result (and its analogue in other theories) is taken to be everything that follows from the Second Theorem. This is not the case: more can be derived from the Second Theorem.

Consider first the specific example of the Lagrangian (19). In addition to the above result (24), we can derive the following from the Second Theorem by using the electromagnetic field equations

$$\frac{\partial L}{\partial A_\mu} - \partial_\nu\left(\frac{\partial L}{\partial(\partial_\nu A_\mu)}\right) = 0. \tag{26}$$

From (23), we conclude that

$$\left[\frac{\partial L}{\partial \psi} - \partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu \psi)}\right)\right](-iq\psi) + \left[\frac{\partial L}{\partial \psi^*} - \partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu \psi^*)}\right)\right] iq\psi^* = 0 \tag{27}$$

and substituting in L_{total} we get

$$\partial_\mu j^\mu = 0 \tag{28}$$

where

$$j^\mu = iq\left(\psi^* D^\mu \psi - \psi D^\mu \psi^*\right) \tag{29}$$

is the familiar electric 4-current. Hence, we see that the current continuity equation can be derived from local gauge symmetry in conjunction with the gauge field equations, via Noether's Second Theorem. The continuity equation can, of course, be derived from the matter field equations, but the Second Theorem shows that while the matter field equations are a sufficient condition for the derivation of the continuity equation, they are not a necessary condition (in the case of Lagrangian (19)).[12] This is in contrast to the case of global gauge symmetry, above, where the current continuity equation associated with the Lagrangian (15) is obtained as a consequence of the matter field equations, via Noether's First Theorem, and where the matter field equations are *necessary* and sufficient for deriving the continuity equation.

What we have here is an instance of the general result that, when the transformations of only the gauge fields depend on $\partial_\mu p_\alpha$, local gauge symmetry plus satisfaction of the gauge field equations leads to a conserved current. In what follows we will see two further methods for arriving at continuity equations such as (28) via local gauge symmetry and satisfaction of the field equations: that of Trautman (in the following section) and that of Utiyama (see section 6).

We end this section with an historical aside. Noether (1918) distinguished between 'improper' and 'proper' conservation laws. 'Improper' conservation laws can be derived without the field equations for the associated field being satisfied. In contrast, where a necessary condition for deriving a conservation law is that the field equations of the associated fields are satisfied, these conservations laws are termed 'proper'. This distinction is due to Hilbert, and was made during considerations of the status of energy conservation in General Relativity (these being what prompted Noether's 1918 work). All the results presented here for local gauge symmetry have analogues in General Relativity, where diffeomorphism invariance is the analogue of local gauge invariance (for further details, see Brading and Brown, 2000). Finally, note also that when Weyl (1918) made the first attempt to derive conservation of charge from a postulated gauge symmetry, independently of Noether's work, his method turned out to be an instance of Noether's Second Theorem with one set of field equations assumed to be satisfied; he later repeated this method (Weyl, 1928 and 1929) in the context of quantum theory (for details, see Brading, 2000).

[12] Bergmann (1959) follows a similar procedure to that presented in this section, without reference to Noether's second theorem. He terms the resulting conservation laws 'strong' conservation laws. These are what Noether called 'improper' conservation laws (see below in the main text). Trautman (1962) appropriates Bergmann's term 'strong' for continuity equations that are satisfied independently of *any* field equations (see section 5).

5 Global Subgroups of Local Gauge Groups

In the case of a theory with local gauge symmetry where there exists a non-trivial global subgroup, we can make use of Noether's First Theorem with respect to this global subgroup in two ways.

First, we can simply apply Noether's First Theorem to global subgroups. In the case of (19), from application of Noether's First Theorem to the global subgroup defined by $\theta =$ constant, with the matter field equations assumed to be satisfied, we obtain (29) as our conserved current once again. Restricting ourselves to the use of Noether's First Theorem in the case of locally gauge symmetric theories is nevertheless subtly misleading, since it suggests that satisfaction of the matter field equations is a necessary condition for the derivation of a conserved current. In fact, as we have seen from Noether's Second Theorem, with respect to (19) the conserved current is an expression of the lack of independence of the matter and gauge fields, and can be obtained by assuming that the gauge field equations are satisfied independently of whether the matter field equations are satisfied. In the locally gauge invariant theory, satisfaction of the matter field equations is merely a sufficient condition for deriving the existence of a conserved current, and not a necessary condition. This can be seen more clearly if we consider the second way of using Noether's First Theorem with respect to a non-trivial global subgroup of a local symmetry group.

Trautman (1962) combines Noether's First Theorem with Noether's Second Theorem in the case where there exists a non-trivial global subgroup defined by

$$\Delta p_a(x^\mu) = \Delta\omega_a. \tag{30}$$

Then (17) becomes

$$\delta\psi_i = a_{\alpha i}\Delta\omega_a + b^\mu_{\alpha i}\partial_\mu\left(\Delta\omega_a\right) = a_{\alpha i}\Delta\omega_a \tag{31}$$

since $\partial_\mu\left(\Delta\omega_a\right) = 0$. Substituting this into the first theorem (10), we get:

$$\sum_i \left[\Psi\right]_i a_{\alpha i} \equiv \partial_\mu j^\mu_\alpha. \tag{32}$$

But from the second theorem (18)

$$\sum_i \left[\Psi\right]_i a_{\alpha i} \equiv \sum_i \partial_\mu\left(\left[\Psi\right]_i b^\mu_{\alpha i}\right)$$

and hence

$$\partial_\mu j^\mu_\alpha \equiv \sum_i \partial_\mu\left(\left[\Psi\right]_i b^\mu_{\alpha i}\right) \tag{33}$$

therefore

$$\partial_\mu \left\{ j^\mu_\alpha - \sum_i \left([\Psi]_i \, b^\mu_{\alpha i} \right) \right\} \equiv 0. \tag{34}$$

Trautman calls such expressions 'strong' conservation laws because they are derived independently of any equations of motion.[13]

In the case of the Lagrangian (19), (34) yields

$$0 \equiv \partial_\mu \left\{ j^\mu - \partial_\nu \left[\frac{\partial L}{\partial A_\mu} - \partial_\nu \left(\frac{\partial L}{\partial \left(\partial_\nu A_\mu \right)} \right) \right] \delta^\nu_\mu \right\}. \tag{35}$$

Hence,

$$\partial_\mu \partial_\nu F^{\mu\nu} \equiv 0. \tag{36}$$

Or, returning to (35), if the gauge field equations are satisfied, we have the conclusion that

$$\partial_\mu j^\mu = 0.$$

These results are obtainable by substituting (19) directly into the Second Theorem, as we have seen. We mention Trautman's result here in part because it makes vivid the point that, in the case of the locally gauge invariant Lagrangian (19), the matter field equations are not a necessary condition for deriving the existence of a conserved matter-field current j^μ: the continuity equation can be derived via (35) by assuming that the gauge field equations are satisfied.

We will see in the following section that (34) is derivable without the assumption that there exists a non-trivial global subgroup.

6 Local Gauge Symmetry: Theorem 3

In this section, we present results due to Utiyama in the form of a theorem; we set this in the context of the results already described in this paper, and draw attention to a corollary due to Al-Kuwari and Taha that we consider to be of particular interest, namely the derivation of the 'Coupled Field Equations' in their general form.

We have seen that we cannot follow the procedure used for global gauge symmetries in the case of local gauge symmetries to form gauge-independent

[13]Trautman takes the term from Bergmann (1949) but, as noted above in a footnote, Bergmann applies the term 'strong' to conservation laws for which the field equations *of the fields associated with the conservation law* are not necessary for the conservation law, even though other field equations are necessarily assumed to be satisfied as part of the derivation.

currents. A current that is dependent on Δp_α is not satisfactory - in particular, such gauge-dependent quantities are not observable.[14] A question that Noether did not address is whether useful, gauge-independent results can be derived from considering the boundary contribution to VP in the case of local symmetries. Al-Kuwari and Taha (1991) consider just this problem, drawing heavily on the work of Utiyama (1956), and citing Frampton's (1987) discussion of Utiyama. No reference to Noether's Second Theorem is made in any of these cases. Utiyama (1956, 1959) starts by retaining both the interior and boundary contributions to VP, and here we follow this more general approach. The Al-Kuwari and Taha results arise when we add the assumption that the Euler-Lagrange equations are satisfied for all the fields on which the Lagrangian depends, as will be indicated.

Theorem 3 *If the action S is invariant under an infinite dimensional continuous Lie group depending smoothly on ρ arbitrary functions $p_\alpha(x^\mu)$ ($\alpha = 1,\ 2, ..., \rho$) and their first derivatives, then there exist three sets of ρ relationships:*

$$\sum_i [\Psi]_i \, a_{\alpha i} \equiv -\sum_i \partial_\mu \left(\frac{\partial L}{\partial \left(\partial_\mu \psi_i\right)} a_{\alpha i} \right) \tag{37}$$

$$\sum_i [\Psi]_i \, b^\mu_{\alpha i} \equiv -\sum_i \left[\frac{\partial L}{\partial \left(\partial_\mu \psi_i\right)} a_{\alpha i} + \partial_\mu \left(\frac{\partial L}{\partial \left(\partial_\mu \psi_i\right)} b^\mu_{\alpha i} \right) \right] \tag{38}$$

$$0 \equiv \sum_i \left[\frac{\partial L}{\partial \left(\partial_\mu \psi_i\right)} b^\mu_{\alpha i} + \frac{\partial L}{\partial \left(\partial_\nu \psi_i\right)} b^\nu_{\alpha i} \right]. \tag{39}$$

Notice that in the special case $p_\alpha(x^\mu) = \omega_\alpha$, (37) reduces to (10), and we recover Noether's First Theorem.

The derivation of Theorem 3 proceeds as follows. We begin from the general expression (5) given in section 2 (the common starting point of Noether's two theorems), and we substitute (17) into this via (6), yielding the expressions

$$\sum_i [\Psi]_i \left(a_{\alpha i} \Delta p_\alpha + b^\mu_{\alpha i} \partial_\mu \left(\Delta p_\alpha\right)\right) \equiv -\sum_i \partial_\mu \left\{ \frac{\partial L}{\partial \left(\partial_\mu \psi_i\right)} \left(a_{\alpha i} \Delta p_\alpha + b^\mu_{\alpha i} \partial_\mu \left(\Delta p_\alpha\right)\right) \right\}, \tag{40}$$

[14]How to deal with this problem is the main subject of the exchange cited in the Introduction, between Karatas and Kowalski (1990) and Al-Kuwari and Taha (1991). Not everyone agrees that gauge-dependent quantities are problematic, however; see for example Bak, Cangemi and Jackiw (1994).

one for every ρ independent arbitrary functions on which the symmetry group depends. For each of these ρ expressions, we proceed by collecting terms in Δp_α and its derivatives:

$$\begin{aligned}
& \sum_i [\Psi]_i \, a_{\alpha i} \Delta p_\alpha + \sum_i [\Psi]_i \, b^\mu_{\alpha i} \partial_\mu (\Delta p_\alpha) \qquad (41) \\
\equiv \quad & -\sum_i \partial_\mu \left(\frac{\partial L}{\partial (\partial_\mu \psi_i)} a_{\alpha i} \right) \Delta p_\alpha \\
& -\sum_i \left[\frac{\partial L}{\partial (\partial_\mu \psi_i)} a_{\alpha i} + \partial_\mu \left(\frac{\partial L}{\partial (\partial_\mu \psi_i)} b^\mu_{\alpha i} \right) \right] \partial_\mu (\Delta p_\alpha) \\
& -\sum_i \left[\frac{\partial L}{\partial (\partial_\mu \psi_i)} b^\mu_{\alpha i} + \frac{\partial L}{\partial (\partial_\nu \psi_i)} b^\nu_{\alpha i} \right] \partial_\nu \partial_\mu (\Delta p_\alpha) .
\end{aligned}$$

But Δp_α and its derivatives are arbitrary, and hence the coefficients of Δp_α and its derivatives must vanish independently, enabling us to extract three separate equations and formulate Theorem 3.

Comparing equations (37), (38) and (39) with those of Utiyama (1959, p. 24), we see that his second and third results are simply (38) and (39), but his first result is different. Utiyama's (1959, p. 24) results are obtained by observing that the interior and boundary contributions must vanish independently,[15] and by focusing on the boundary contribution. Noether's Second Theorem is the condition for the vanishing of the interior, and we can substitute (18) into (37) to obtain Utiyama's first result (1959, p. 24, equation 2.6):

$$\sum_i \partial_\mu \left(\frac{\partial L}{\partial (\partial_\mu \psi_i)} a_{\alpha i} + [\Psi]_i \, b^\mu_{\alpha i} \right) \equiv 0. \qquad (42)$$

The significance of the results (37) and (42) can be understood as follows. Consider the special case where the fields ψ_i on which the Lagrangian depends divide into two sets: one whose gauge transformations depend on p_α but not on $\partial_\mu p_\alpha$, the other whose transformations depend on $\partial_\mu p_\alpha$ but not on p_α. Then it follows from (37) and (42) that if either set of field equations is satisfied, there exists a conserved current of the form

$$j^\mu_\alpha := -\sum_i \frac{\partial L}{\partial (\partial_\mu \psi_i)} a_{\alpha i}. \qquad (43)$$

The Lagrangian (19) is just such a special case: as discussed in section 4, above, the continuity equation for the electric current can be derived from satisfaction of either the matter field equations or the gauge field equations.

[15]This is because the functions p_α are arbitrary, and so we could choose that the p_α and their derivatives vanish on the boundary. Therefore, the interior contribution must vanish independently of what happens on the boundary, and since the entire variation must vanish in all cases, the boundary contribution must vanish even when the arbitrary functions are not chosen to vanish on the boundary.

Equation (42) is the result obtained by Trautman (when we substitute in (43)) via consideration of the global subgroup of the local gauge group $p_\alpha(x^\mu) = \omega_\alpha$ (see section 5, above): we see here that Trautman's result is derivable more generally.

Turning now to equation (38), we first note Utiyama's (1959, p. 27, equation 2.14) observation that in the specific case of the Lagrangian (19) we obtain:

$$\sum_i \frac{\partial L}{\partial\left(\partial_\mu \psi_i\right)} a_{\alpha i} \equiv \frac{\partial L}{\partial A_\mu}. \tag{44}$$

Thus, the current associated with the matter fields equations (on the left-hand side of (44)) is identified with the current of Maxwell's equations with sources (on the right-hand side of (44)). In more general terms, when condition (44) is satisfied the matter-field current associated with the Lagrangian acts as the source for the gauge fields.

There is, however a wider and completely general significance to (38), which we turn to in the following section.

The significance of the final set of equations presented in Theorem 3, (39), is seen most clearly from the specific example of the Lagrangian (19). In this case (see Utiyama, 1959, p. 27), (39) says that the derivative $\partial_\nu A_\mu$ must appear in the Lagrangian in the combination $\partial_\nu A_\mu - \partial_\mu A_\nu$, or in other words (24). More generally, we have a restriction on those fields whose transformations depend on $\partial_\mu p_\alpha$.

The three sets of equations given by Al-Kuwari and Taha (1991, equations 34), modulo some technical details, are arrived at from (37), (38) and (39) by assuming that the Euler-Lagrange equations are satisfied for all the fields on which the Lagrangian depends, i.e. $\sum_i [\Psi]_i = 0$, so that the left-hand sides of (37) and (38) vanish.[16] In our opinion, their second set of equations is their most important result. We discuss this in detail in the following section, but first some brief remarks about the other two sets of equations. Their first set of equations is potentially misleading with respect to the necessary and sufficient conditions

[16] At this point, it is perhaps worth offering the following cautionary remark. Throughout this paper we have been careful to write that a sufficient condition for $\sum_i [\Psi]_i$ vanishing is that the Euler-Lagrange equations are satisfied *for all the fields on which the Lagrangian depends*. More precisely still, we restrict our attention to those fields that feature in the symmetry transformation under consideration. In the presence of 'background fields' or 'absolute objects' that participate in the symmetry transformation, satisfaction of the Euler-Lagrange equations may not be sufficient for the vanishing of all the Lagrange expressions $[\Psi]_i$, and hence $\sum_i [\Psi]_i$ may not be zero. For example, consider the locally gauge invariant Lagrangian

$$L_1(\psi, \partial_\mu \psi, \psi^*, \partial_\mu \psi^*, A_\mu, x^\mu) = D_\mu \psi D^\mu \psi^* - m^2 \psi \psi^*.$$

In this case, $[\Psi]_{A_\mu} = \frac{\partial L_1}{\partial A_\mu} - 0 = j^\mu$, and the left-hand side of equation (38) does not vanish even when all the Euler-Lagrange equations associated with L_1 are satisfied. Failure to notice this would lead to inconsistent results.

for deriving the existence of the conserved current: their derivation assumes satisfaction of *both* the matter field equations and the gauge field equations, but satisfaction of *either* set of field equations is sufficient for the conserved current to be derived, as we saw from Noether's Second Theorem in section 4 above. Nevertheless, as Al-Kuwari and Taha emphasise, the first set of equations does make clear the important point that no further conserved currents can be derived from local gauge symmetry than from global gauge symmetry when the Euler-Lagrange equations are assumed to be satisfied. The third set of equations is also potentially misleading: Al-Kuwari and Taha assume from the outset that the Euler-Lagrange equations are satisfied, so their version of (39) appears to depend on the satisfaction of these equations; recall, however, that (39) can be derived from the local gauge invariance of the Lagrangian independently of any Euler-Lagrange equations, as we also saw from Noether's Second Theorem in section 4 above.

6.1 Coupled Field Equations

To bring out the wider significance of (38) we turn to Al-Kuwari and Taha.[17] When the Euler-Lagrange equations are assumed to be satisfied for all the fields on which the Lagrangian depends (or, more precisely, for all the fields whose transformations depend on $\partial_\mu p_\alpha$), equation (38) of Theorem 3 yields what we may call the **'Coupled Field Equations'**:

$$j^\mu_\alpha = \sum_i \partial_\sigma \left(F^{i\mu} b^\sigma_{\alpha i} \right) \tag{45}$$

where

$$j^\mu_\alpha := - \sum_i \frac{\partial L}{\partial \left(\partial_\mu \psi_i \right)} a_{\alpha i} \tag{46}$$

and

$$F^{i\mu} := \frac{\partial L}{\partial \left(\partial_\mu \psi_i \right)}. \tag{47}$$

We term (45) 'Coupled Field Equations' because of the form of the interrelationship they describe between the different fields appearing in the Lagrangian. In the specific example we have been considering, the Lagrangian (19), (45) becomes

$$j^\mu = \partial_\nu F^{\nu\mu} \tag{48}$$

[17] Although consistent with the spirit of Al-Kuwari and Taha's results, the discussion presented here differs from theirs in various technical respects.

where

$$F^{\nu\mu} = \frac{\partial L}{\partial\left(\partial_\mu A_\nu\right)}. \tag{49}$$

Condition (48) tells us that the vanishing of the boundary contribution to the variation in the action requires a balance between the current associated with the matter fields and the propagation of the gauge fields.

Notice the important point that the general form of these coupled field equations, (45), has been derived independently of the form of any specific Lagrangian or Euler-Lagrange equations. We simply assume that the Lagrangian is invariant under local gauge transformations of the general form (17), and that the field equations are satisfied, but we don't have to know what the field equations are in order to derive the general form of the coupled field equations.

This concludes the results derivable from Noether's variational problem for global and local gauge symmetries.[18]

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[18]The results are straightforwardly generalisable to the case where $\delta x \neq 0$ (see Doughty, 1990; Utiyama, 1959; and Brading and Brown, 2000).

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